MODELING VACCINATION AND TREATMENT OF HIV/ AIDS EPIDEMICS IN AN AGE - STRUCTURED POPULATION

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Abstract

We proposed a deterministic age- structured vaccination and treatment model ofHIVand AIDS, along the line ofMcCamy - Foester age-structured pop ulation model, with structured population compartments and class dependent activity level (interaction functions), per capital force of infection, natural mortality rate and disease induced death rate . The models equations are reduced to ordi nary differential equations. Their steady state solutions are obtained and examined for local stability. The disease-free state is found to be stable if vaccination md removal rates are simultaneously maximized. This will lead to a corresponding decrease in the size of HIV -infected individuals and the number of infected cases progressing to AIDS.

Keywords: *Vaccination, treatment, HIV/AIDS, population.*

INTRODUCTION

Expositions on Biological aspects and transmission mechanism of HIV/ AIDS can be found in Hethcote it *al.* (1992), Inaba (1990), Castillo-Chavez (1991), Angel *et al*, (1995), Kimbir *et al.* (2001), Akinwande (2006) and Umar (2007). However in this work, we proposed an age-structured deterministic model of HTV/AIDS along the line of Mckendrick-Foerster type S-I-R age-structured epidemic model using Inaba (1990), Hethcote *et al.* (1992) and Castillo-Chavez *et al*. (1991) representations of proportionate mixing type per capital force of infection and activity level. We assumed constant recruitment contrary to Akinwande (2006) and a general mixing or interaction function (activity level) along the line of Castillo-Chavez (1991), Hethcote *et al* (1992) and Mugisha *et al* (2003) without vertical transmission of the disease. Clinical therapies are now provided to pregnant mothers to help reduce transmission of the vims to thuir new bom and so vertical transmission is no longer considered a major source of transmission of the virus itt mOSt developing countries, like the heterosexual mode. This work therefore focuses on the heterosexual mode of transmission of IITV/AIDS.

Model Assumptions

We assume the following,

(i) The disease is transmitted through heterosexual mode, based on proportionate mixing, between susceptible of age and an infectious individuals of age a[1].

(ii) Negligible latency period for the infection

(iii) An infectious individual remains in the compartment for a period

Model Formulation

Using assumptions (i) - (iv) and the representation for the vital rates, the dynamics of the population densities are obtained as,

$$\frac{\partial S(t,a)}{\partial t}+\frac{\partial S(t,a)}{\partial a}=-\mu(a)S(t,a)-\lambda(t,a)S(t,a) \tag{1}$$

$$\frac{\partial I(t,a)}{\partial t}+\frac{\partial I(t,a)}{\partial a}=\lambda(t,a)S(t,a)-\mu(a)I(t,a)-\alpha(\tau)I(t,a)-\gamma(a)I(t,a) \tag{2}$$

$$\frac{\partial R(t,a)}{\partial t}+\frac{\partial R(t,a)}{\partial a}=\gamma(a)I(t,a)-\alpha'(\tau)R(t,a) \tag{3}$$

$$S(t,0)=\Delta,\ \ I(t,0)=0,\ \ \ S(0,a)=S_0(a)\ ,\ I(0,a)=I_0(a),,\ R(0)=0 \tag{4}$$

$$I(t,a)=\lambda(t,a)S(t,a)$$

$$\lambda(t,a)=\frac{1}{N(t)}\int_0^\infty k(a,\bar{a})I(t,\bar{a})d\bar{a} \tag{5}$$

$$N(t)=\int_0^\infty n(t,a)da \tag{6}$$

Where $n(t,a)=S(t,a)+I(t,a)+R(t,a)$

$$S(t)=\int_0^\infty S(t,a)da\ ,\ \mathrm{R(t)}=\int_0^\infty \mathrm{R(t,}a)\mathrm{d}\tau\ ,\quad \mathrm{I(}t)=\int_0^\infty \mathrm{I(t,}a)\mathrm{d}\tau\ , \tag{7}$$

where μ (a) is the natural mortality rate, $\lambda(t,a)$ is time and age dependent force of infection, $\alpha(\tau)$ is the infection time dependent disease induced death rate, $\gamma(a)$ is the rate at whic h individuals in the infective population compartment are removed for treatment. The activity level between a susceptible of age a and an infectious of age $\bar{a}$ is $k(a,\bar{a})$. Suppose at the beginning of the infection the population is in a steady demographic state , (steady state age distribution, independent of time), with equilibrium age density,

$$N(a)=NV(a), \tag{8}$$

where $V(a)=\exp(-\int_0^a \mu(a)da)$

is the survival function, defined as the probability that an individual susceptible survive to age a. We reduce equations, (1) - (3), to a two age-group HIV/AIDS transmission model defined within the age range $[a_{i-1}, a_i]$, i = 1, 2, in which both groups are assumed sexually active, and productive. The numbers of susceptible, infected and removed individuals in the ith age group can be represented by the following,

$$S_i(t)=\int_{a_{i-1}}^{a_i} S(t,a)da,\qquad \mathrm{I}_i(t)=\int_{a_{i-1}}^{a_i} I(t,a)da\qquad \mathrm{R}_i(t)=\int_{a_{i-1}}^{a_i} R(t,a)da$$

Suppose at the start of the epidemic the population is at the stationary age distribution with exponential growth in all the classes as,

$$N(t,a)=e^{\omega_0 t}\psi(a)\ .$$

Where ω_0 is the intrinsic rate of population growth at the steady state, and $\psi(a)$ is the t otal population size at age a. T he intrin sic population growth rate can be obtained from the Lotka characteristics equation,

$$\int_0^{\infty} e^{-\omega_0 a} f(a) l(a) da = 1 \tag{9}$$

Where $f(a)$ and $l(a)$ are the fertility and survival functions respectively.

The total number of individuals in the age interval $[a_{i-1} a_i)$ is,

$$N_i(t) = \int_{a_{i-1}}^{a_i} N(t,a)da = e^{\omega_0 t}\int_{a_{i-1}}^{a_i} \psi(a)da = e^{\omega_0 t} m_i \tag{10}$$

Let N (t) denotes the total population size. Then $N(t)$ satisfies,

$$N(t) = \sum_{j=1}^{2} (S_j(t) + I_j(t)) = \theta e^{\omega_0 t}, \quad \theta = m_1 + m_2 \tag{11}$$

Where $m_i = \int_{a_{i-1}}^{a_i} \psi(a)da$ is the size of the ith age-group at the steady state at time t = 0

We approximate $m_i = \psi(a)(a_i - a_{a-1})$ in line with Mugisha *et al*, (2003) , Castillo-Chavez (1991) and Hethcote *et al* , (1992) and also consider $a_{i-1} \le a \le a_i$, with the following parameters as class depended,

$$\mu(a) = \mu_i, \; k(a) = k_i, \; \alpha(a) = \alpha_i \; \gamma(a) = \gamma_i \; \lambda(t,a) = \lambda_i \tag{12}$$

The renewal equation for the susceptible is assumed constant as $S(t,0) = \Delta$, with that of infective $I(t,0) = 0$.

Also, suppose the transfer rate constant between the two age groups is defined by,

$$c_i = \frac{1}{a_i - a_{i-1}} \quad , \; i = 1, 2 \tag{13}$$

Let the transition of individuals from one epidemiological group into another be described by following equations,

$$S(t_i, a_i) = c_i S_i(t), \; , I(t_i, a_i) = c_i I_i(t), \quad R(t_i, a_i) = c_i R_i(t), \; \psi(a_i) = c_i m_i \tag{14}$$

The population fractions of the susceptible, infective and removed compartments are given by,

$$s_i(t) = \frac{S_i(t)}{N_i(t)} = \frac{\int_{a_{i-1}}^{a_i} S(t,a)da}{\int_{a_{-1}}^{a_i} n(t,a)da} = \frac{\int_{a_{-1}}^{a_i} S(t,a)da}{e^{\omega_0 t} m_i} \tag{15}$$

$$i_i(t) = \frac{\int_{a_{i-1}}^{a_i} I(t,a)da}{\int n(t,a)da} = \frac{\int_{a-1}^{a} I(t,a)da}{e^{\omega_0 t} m} \tag{16}$$

$$r_i(t) = \frac{R_i(t)}{N_i(t)} = \frac{\int_{a_{i-1}}^{a_i} R(t,a)da}{e^{\omega t} m} \tag{17}$$

Where $s_i(t) + i_i(t) + r_i(t) = 1$,

If $r_i(t) = 0$, the model reduces to the standard simple S - I epidemic model with $s_i(t) + i_i(t) = 1$, and no preventive therapy is adopted by the population. However we assume $r_i(t) \neq 0$ and a constant class-dependent contact or interaction function, between a susceptible in the ith age group and an infective in the jth age group described by the following,

$$k(a,\bar{a}) = k_{ij} \quad \text{, for } a \in [a_{i-1}, a_i],\ \bar{a} \in [a_{j-1}, a_j] \tag{18}$$

The class-dependent per capital force of infection has the form,

$$\lambda_i(t) = \frac{\sum_{k=1}^{2} k_{ik} I_k(t)}{\sum_{i=1}^{2} N_i(t)} = \frac{\sum_{k=1}^{2} k_{ik} e^{\omega_0 t} m_k i_k(t)}{\sum_{i=1}^{2} e^{\omega_0 t} m_i} = \sum_{k=1}^{2} k_{ik} m_k i_k(t) \tag{19}$$

For i = 1, 2 we have the following expressions for the class dependent per capital force of infection,

$$\lambda_i(t) = \sum_{j=1}^{2} k_{ij} m_j i_j,\ \text{i, j} = 1,2 \tag{20}$$

Integrating equation (1) - (3) with respect to a over (a_{i-1}, a_i) and also taking cognizance of equations (14) and (18), we get,

$$S(t,a_i) - S(t,a_{i-1}) + \frac{dS_i(t)}{dt} = -\lambda_i S_i(t) - \mu_i S_i(t)$$

When i = 1, we have,

$$\frac{ds_1(t)}{dt} = \Delta - (c_1 + \omega_0 + \mu_1 + \lambda_1) s_1(t)$$

When i = 2, we have,

$$\frac{ds_2(t)}{dt} = c_1 \frac{m_1}{m_2} s_1(t) - (\omega_0 + c_2 + \mu_2 + \lambda_2) s_2(t),$$

Also integrating equation (2) with respect to a over (a_{i-1}, a_i) gives the equation, (21)

$$\frac{dI_i(t)}{dt} + I(t,a_i) - I(t,a_{i-1}) = \lambda_i S_i(t) - \mu_i I_i(t) - \alpha_i I_i(t) - \gamma_i I(t)$$

When i = 1, we have

$$\frac{di_1(t)}{dt} = \lambda_1 s_1(t) - (\omega_0 + \mu_1 + c_1 + \alpha_1 + \gamma_1) i_1(t)$$

When i = 2 we have,

$$\frac{di_2(t)}{dt} = \frac{m_1}{m_2} + \lambda_2 s_2(t) - (\mu_2 + \alpha_2 + c_2 + \gamma_2 + \omega_0) i_2(t)$$

When i = 1 we have,

$$\frac{dr_1(t)}{dt} + (\omega_0 + c_1 + \alpha'_1) r_1(t) = \gamma_1 i_1(t)$$

When i =2 we have,

$$\frac{dr_2(t)}{dt} + (\omega_0 + c_2 + \alpha'_2)r_2(t) = c_1 r_1(t) + \gamma_2 i_2(t)$$

The two age groups HIV and AIDS epidemic model with treatment is described by following dynamic equations,

$$\frac{ds_i(t)}{dt} = \Delta - (c_1 + \omega_0 + \mu_1 + \lambda_1)s_1(t) \tag{22}$$

$$\frac{ds_2(t)}{dt} = c_1 \frac{m_1}{m_2} s_1(t) - (\omega_0 + c_2 + \mu_2 + \lambda_2)s_2(t) \tag{23}$$

$$\frac{di_1(t)}{dt} = \lambda_1 s_1(t) - (\omega_0 + \mu_1 + c_1 + \alpha_1 + \gamma_1)i_1(t) \tag{24}$$

$$\frac{di_2(t)}{dt} = \frac{m_1}{m_2} + \lambda_2 s_2(t) - (\mu_2 + \alpha_2 + c_2 + \gamma_2 + \omega_0)i_2(t)$$

$$\frac{dr_1(t)}{dt} = -(\omega_0 + c_1 + \alpha'_1)r_1(t) + \gamma_1 i_1(t)$$

$$\frac{dr_2(t)}{dt} = -(\omega_0 + c_2 + \alpha'_2)r_2(t) + c_1 r_1(t) + \gamma_2 i_2(t) \tag{25}$$

Where,

$$s_1(t) = \frac{\Delta}{\eta_1} + \psi e^{-\eta_1 t}, \quad \eta_1 = c_1 + \omega_0 + \mu_1 + \lambda_1, \ \psi = s_1(0) - \frac{\Delta}{\eta_1} \tag{26}$$

$$s_2(t) = \frac{c_1 m_1 \Delta}{m_2 \eta_1 \eta_2} - \frac{c_1 p_1 \psi}{m_2(\eta_1 - \eta_2)} e^{-\eta_1 t} + k e^{-\eta_2 t}, \ k = s_2(0) - \frac{c_1 m_1}{m_2}\left(\frac{\Delta}{\eta_1 \eta_2} - \frac{\psi}{\eta_1 - \eta_2}\right) \tag{27}$$

$$i_1(t) = M_1 - M_2 e^{-\eta_1 t} + M e^{-\eta_3 t}, \ M = i_1(0) - \frac{\lambda_1 \Delta}{\eta_1 \eta_3} + \frac{\psi}{\eta_1 - \eta_3}$$

$$M_1 = \frac{\lambda_1 \Delta}{\eta_1 \eta_3}, \quad M_2 = \frac{\psi}{\eta_1 - \eta_3} \tag{28}$$

$$i_2(t) = D_1 + D_2 e^{-\eta_1 t} - D_3 e^{-\eta_2 t} + D_4 e^{-\eta_4 t} \tag{29}$$

$$D_1 = \frac{m_1}{m_2\eta_4} + \frac{\lambda_2 c_1 \Delta}{m_2\eta_1\eta_2\eta_4},\quad D_2 = \frac{\lambda_2 c_1 m_1 \psi}{m_2(\eta_2 - \eta_1)(\eta_1 - \eta_4)},\quad D_3 = \frac{k}{\eta_2 - \eta_4}$$

$$D_4 = i_2(0) - \left(\frac{m_1}{m_2\eta_4} + \frac{\lambda_2 c_1 m_1 \Delta}{m_2\eta_1\eta_2\eta_4} + \frac{\lambda_2 c_1 m_1 \psi}{m_2(\eta_1 - \eta_2)(\eta_1 - \eta_4)} - \frac{k}{\eta_2 - \eta_4}\right)$$

$$\eta_4 = \alpha_2 + \mu_2 + c_2 + \gamma_2 + \omega_0, \quad \eta_3 = \alpha_1 + \mu_1 + c_1 + \omega_0 + \gamma_1$$

$$\eta_2 = \omega_0 + c_2 + \mu_2 + \lambda_2$$

$$r_1(t) = B_1 + B_2 e^{-\eta_5 t} - Me^{-\eta_3 t} + B_3 e^{-\eta_5 t} \tag{30}$$

$$B_1 = \frac{\gamma_1 M_1}{\eta_5},\quad B_2 = \frac{\gamma_1 M_2}{\eta_1 - \eta_5},\quad B_3 = r_1(0) - \frac{\gamma_1 M_1}{\eta_5} - \frac{\gamma_1 M_2}{\eta_1 - \eta_5} + M$$

$$\eta_5 = c_1 + \alpha'_1 + \omega_0$$

$$r_2(t) = E_1 - E_2 e^{-\eta_1 t} + E_3 e^{-\eta_2 t} + E_4 e^{-\eta_3 t} + E_5 e^{-\eta_4 t} + E_5 e^{-\eta_6 t} \tag{31}$$

$$E_1 = \frac{1}{\eta_6}(c_1 B_1 + \gamma_2 \Delta),\quad E_2 = \frac{1}{\eta_1 - \eta_6}(c_1 B_2 + B_3 - D_2),\quad E_3 = \frac{D_3}{\eta_2 - \eta_6},\quad E_4 = \frac{M}{\eta_3 - \eta_6},$$

$$E_5 = \frac{D_4}{\eta_4 - \eta_6},\quad E_6 = r_2(0) - \frac{1}{\eta_6}(c_1 B_1 + \gamma_2 \Delta) + \frac{1}{\eta_1 - \eta_6}(c_1 B_2 + B_3 + D_2) - \frac{M}{\eta_3 - \eta_6} - \frac{D_4}{\eta_4 - \eta_6}$$

$$\eta_6 = C_2 + \alpha'_2 + \omega_0$$

Hence for sufficiently large values of t, we get the following equations,

$$s_1(t) = \frac{\Lambda}{\eta_1},\quad s_2(t) = \frac{c_1 m_1 \Delta}{m_2\eta_1\eta_2},\quad i_1(t) = \frac{\lambda_1 \Delta}{\eta_1\eta_3},\quad i_2(t) = \frac{m_1}{m_2\eta_4} + \frac{\lambda_2 c_1 \Delta}{m_2\eta_1\eta_2\eta_4},$$

$$r_1(t) = \frac{\gamma_1\lambda_1\Delta}{\eta_5\eta_1\eta_3},\quad r_2(t) = \frac{1}{\eta_6}(c_1 B_1 + \gamma_2 \Delta) = \frac{1}{\eta_6}\left(\frac{c_1\lambda_1\gamma_1}{\eta_1\eta_3\eta_5} + \gamma_2\right)\Delta \tag{32}$$

Since η_3 and η_4 are dependent on removal rates the disease compartments, $i_1(t)$ and $i_2(t)$ for both groups will be significant reduced if considerable numbers of the infectives are treatment and hence AIDS defining disease will also be reduced with a corresponding reduction in AIDS related death. The assumption of treatment therapies has significant impart on the transmission pattern of the virus.

Existence and stability of the disease - free steady States.

The Disease free state is given by,

$$E_0 = (s_1^*, s_2^*, 0,0,0,0) = \left(\frac{\Delta}{\eta_1}, c_1 \frac{m_1 \Delta}{m_2 \eta_1 \eta_2}, 0,0,0,0\right)$$

With the Jacobian matrix at the disease- free state as,

$$J_{E_0} = \begin{pmatrix} -\eta_1 & 0 & 0 & 0 & 0 & 0 \\ c_1 \frac{m_1}{m_2} & -\eta_2 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & -\eta_3 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & -\eta_4 & 0 & 0 \\ 0 & 0 & \gamma_1 & 0 & -\eta_5 & 0 \\ 0 & 0 & 0 & \gamma_2 & c_1 & -\eta_6 \end{pmatrix}$$

Where $(\eta_1 + \kappa_1)(\eta_2 + \kappa_2)(\eta_3 + \kappa_3)(\eta_4 + \kappa_4) \det \begin{pmatrix} -(\eta_5 + \kappa_5) & 0 \\ c_1 & -(\eta_6 + \kappa_6) \end{pmatrix} = 0$

The characteristic equation is,

$$(\eta_1 + \kappa_1)(\eta_2 + \kappa_2)(\eta_3 + \kappa_3)(\eta_4 + \kappa_4)(\eta_5 + \kappa_5)(\eta_6 + \kappa_6) = 0 \qquad (33)$$

$$\kappa_k = -\eta_k, \; k = 1, 2, \ldots, 6$$

Since the model parameters are non-negative, $\eta_k > 0$ for all $k = 1, \ldots, 6$ and so all solutions of the characteristic equation have negative real part, a requirement for locally asymptotically stable steady disease state and so the disease-free steady state would be locally asymptotically stable if treatment is adopted by a fraction of the infected population.

However, the basic reproductive number R_0 without vaccination can also be obtained using the next generation matrix in which R_0 is defined as the spectral radius (dominant eigenvalue) of the next generation operator obtained from matrix FV^{-1}. F_i, and V_i, are defined by,

$$F = \left| \frac{\partial F_i(E_0)}{\partial x_j} \right| , \quad V = \left| \frac{\partial V_i(E_0)}{\partial x_j} \right|$$

Where $j = 1, 2, \ldots, m$ are the infected compartments, E_0 is the disease-free equilibrium state. $F_i(\bar{E})$ is the appearance of new infections in compartment i and $V_i = V_i^-(\bar{E}) - V_i^+(\bar{E})$, V_i^+ is the rate of transfer of individual into compartment i, and V_i^- is the rate of transfer of individual out compartment i, $\bar{E} = E_i$, and denotes the proportion of individuals in the i compartment. The matrices F and V are,

$$F = \begin{pmatrix} \frac{\Delta k_{11} m_1}{\eta_1} & k_{12} m_2 \frac{\Delta}{\eta_1} & 0 & 0 \\ k_{21} m^2{}_1 \frac{c_1 \Delta}{m_2 \eta_1 \eta_2} & k_{22} \frac{c_1 m_1 \Delta}{\eta_1 \eta_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V^{-1} = \begin{pmatrix} \frac{1}{\eta_3} & 0 & 0 & 0 \\ 0 & \frac{1}{\eta_4} & 0 & 0 \\ -\frac{\gamma}{\eta_3 \eta_5} & 0 & \frac{1}{\eta_5} & 0 \\ 0 & \frac{\gamma_2}{\eta_4 \eta_6} & 0 & \frac{1}{\eta_6} \end{pmatrix}$$

$$FV^{-1} = \begin{pmatrix} \frac{\Delta k_{11} m_1}{\eta_3 \eta_1} & k_{12} m_2 \frac{\Delta}{\eta_1 \eta_4} & 0 & 0 \\ k_{21} m^2{}_1 \frac{c_1 \Delta}{m_2 \eta_1 \eta_3} & k_{22} \frac{c_1 m_1 \Delta}{\eta_1 \eta_2 \eta_4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The characteristic equation is,

$$\kappa^2(\kappa^2 - (A_1 + B_2)\kappa - (A_2 B_1 - A_1 B_2) = 0,\ A_1 = \frac{k_{11} m_1 \Delta}{\eta_1 \eta_3},\ A_2 = k_{12} m_2 \frac{\Delta}{\eta_1 \eta_4}$$

$$B_1 = k_{21} m_1{}^2 \frac{c_1 \Delta}{m_2 \eta_1 \eta_3},\quad B_2 = k_{22} m_1 \frac{c_1 \Delta}{\eta_1 \eta_2 \eta_4}$$

The non zero eigenvalues are,

$$\kappa_1 = \frac{m_1 \Delta}{2\eta_1}\left(\frac{k_{11}}{\eta_3} + \frac{k_{22}}{\eta_2 \eta_4}\right) + \frac{1}{2}\sqrt{\left(\frac{m_1 \Delta}{\eta_1}\right)^2 \left(\frac{k_{11}}{\eta_3} + \frac{k_{22}}{\eta_2 \eta_4}\right)^2 + 4\frac{m_1 \Delta^2 c_1 k_{11}}{\eta_1{}^2 \eta_3 \eta_4}\left(k_{21} m_1 - \frac{k_{22} m_2}{\eta_2}\right)}$$

$$\kappa_2 = \frac{m_1 \Delta}{2\eta_1}\left(\frac{k_{11}}{\eta_3} + \frac{k_{22}}{\eta_2 \eta_4}\right) - \frac{1}{2}\sqrt{\left(\frac{m_1 \Delta}{\eta_1}\right)^2 \left(\frac{k_{11}}{\eta_3} + \frac{k_{22}}{\eta_2 \eta_4}\right)^2 + 4\frac{m_1 \Delta^2 c_1}{\eta_1{}^2 \eta_3 \eta_4}\left(k_{11} k_{21} m_1 - \frac{k_{11} k_{22} m_2}{\eta_2}\right)}$$

$$R_0 = \kappa_1$$

The disease-free state is asymptotically stable if $\kappa_1 < 1$, otherwise the endemic steady state $(s_1^*, s_2^* i_1^*, i_2^*, r_1^*, r_2^*)$ is stable,

$$\text{Where } s_1^* = \frac{\Delta}{\eta_1}, \quad s_2^* = c_2 \frac{m_1 \Delta}{m_2 \eta_1 \eta_2}, \quad i_1^* = \frac{\lambda_1 \Delta}{\eta_1 \eta_3}, \quad i_2^* = \frac{m_1(\lambda_2 \Delta c_1 + \eta_1 \eta_2)}{m_2 \eta_1 \eta_2 \eta_3}$$

$$r_1^* = \frac{\gamma_1 \lambda_1 \Delta}{\eta_1 \eta_3 \eta_5}, \quad r_2^* = \frac{c_1 \lambda_1 \gamma_1 \Delta}{\eta_1 \eta_3 \eta_5 \eta_6} + \frac{\gamma_2 m_1(\lambda_2 \Delta c_1 + \eta_1 \eta_2)}{m_2 \eta_1 \eta_2 \eta_3 \eta_6} \tag{35}$$

Model with Vaccination and Preventive therapies

No cure has been found for HIV / AIDS epidemics. The only available mechanisms of control are the preventive therapies and so we will examine the effects of vaccination on the transmission of HIV / AIDS epidemics as follows.

The proportion of the susceptible population vaccinated is age dependent $\sigma(a)$, with $\beta(t,a)$ as the per capital force of infection for those vaccinated. The vaccine is assumed effective with few cases of transmission and so $\lambda(t,a) > \beta(t,a)$. The fraction of the vaccinated population density is,

$$v_i(t) = \frac{V_i(t)}{N_i(t)} = \frac{\int_{a_{i-1}}^{a_i} V(t,a)da}{\int_{a_{-1}}^{a_i} n(t,a)da} = \frac{\int_{a_{-1}}^{a_i} V(t,a)da}{e^{\omega_0 t} m_i}$$

$$\text{Where } n(t,a) = S(t,a) + I(t,a) + R(t,a) + V(t,a), \quad N(t) = \int_0^\infty n(t,a)da \ , \ \sigma(a) = \sigma_i \ , \ i = 1,2$$

$$V(t,a) = c_i V_i(t) \ , \ V_i(t) = \int_{a_{i-1}}^{a_i} V(t,a)da$$

The dynamics of the vaccinated population compartment is described by,

$$\frac{\partial V(t,a)}{\partial a} + \frac{\partial V(t,a)}{\partial t} = \sigma(a)S(t,a) - \mu(a)V(t,a) - \beta(t,a)V(t,a)$$

$$V(t,0) = 0, \quad V(0,a) = V_0(a) \ , \ \beta(t,a) = \frac{1}{N(t)} \int_0^\infty m(a,\bar{a}) I(t,\bar{a}) d\bar{a} \tag{36}$$

Also equations (1) and (2) now takes the form,

$$\frac{\partial S(t,a)}{\partial t} + \frac{\partial S(t,a)}{\partial a} = -\mu(a)S(t,a) - \lambda(t,a)S(t,a) - \sigma(a)S(t,a)$$

$$\frac{\partial I(t,a)}{\partial t} + \frac{\partial I(t,a)}{\partial a} = \lambda(t,a)S(t,a) + \beta(t,a)V(t,a) - \mu(a)I(t,a) - \alpha(\tau)I(t,a) - \gamma(a)I(t,a) \tag{37}$$

$$S(t,0) = \Delta, \ I(t,0) = 0, \quad S(0,a) = S_0(a) \ , \ I(0,a) = I_0(a),$$

$$I(t,a) = \lambda(t,a)S(t,a)$$

$$\lambda(t,a) = \frac{1}{N(t)} \int_0^\infty k(a,\bar{a}) I(t,\bar{a}) d\bar{a}$$

Since equation (3) is unchanged the problem reduces to solving the following governing equations (37) and (3).

Integrating equations (37) with respect to a over (a_{i-1}, a_i) gives the following,

$$V(t,a_i) - V(t,a_{i-1}) = \frac{dvV(t)}{dt} = \sigma_i S_i(t) - \mu_i V_i(t) - \beta_i V_i(t)$$

i =1 gives, $V(t,a_1) - V(t,a_0) + \frac{dV_1(t)}{dt} = \sigma_1 S_1(t) - \mu_1 V_1(t) - \beta_1 V_1(t)$

$$V(t,a_0) = V(t,0) = 0\,,\ V(t,a_1) = c_1 V_1(t)\,,$$

We have,

$$\frac{dv_1(t)}{dt} = -(c_1 + \mu_1 + \beta_1)v_1(t) + \sigma_1 s_1(t)$$

Similarly when i = 2 we have,

$$\frac{dv_2(t)}{dt} = c_1 v_1(t) - (c_2 + \mu_2 + \beta_2)v_2(t) + \sigma_2 s_2(t)$$

$$\frac{ds_1(t)}{dt} = \Delta - (c_1 + \omega_0 + \mu_1 + \lambda_1 + \sigma_1)s_1(t)$$

$$\frac{ds_2(t)}{dt} = c_1 \frac{m_1}{m_2} s_1(t) - (\omega_0 + c_2 + \mu_2 + \lambda_2 + \sigma_2)s_2(t)$$

$$\frac{di_1(t)}{dt} = \lambda_1 s_1(t) + \beta_1 v_1(t) - (\omega_0 + \mu_1 + c_1 + \alpha_1 + \gamma_1)i_1(t)$$

$$\frac{di_2(t)}{dt} = \frac{m_1}{m_2} + \lambda_2 s_2(t) + \beta_2 v_2(t) - (\mu_2 + \alpha_2 + c_2 + \gamma_2 + \omega_0)i_2(t)$$

$$\frac{dr_1(t)}{dt} = -(\omega_0 + c_1 + \alpha'_1)r_1(t) + \gamma_1 i_1(t)$$

$$\frac{dr_2(t)}{dt} = -(\omega_0 + c_2 + \alpha'_2)r_2(t) + c_1 r_1(t) + \gamma_2 i_2(t)$$

Where,

$$s_1(t) = \frac{\Delta}{\theta_1} + \xi e^{-\theta_1 t} \quad \theta_1 = (c_1 + \omega_0 + \mu_1 + \lambda_1 + \sigma_1),\ \xi = s_1(0) - \frac{\Delta}{\theta_1}\ ,$$

$$s_2(t) = g_1 - g_2 e^{-\theta_1 t} + k e^{-\theta_2 t}\ ,\ k = s_2(0) - \frac{c_1 m_1}{m_2}\left(\frac{\Delta}{\theta_1\theta_2} - \frac{\psi}{\theta_1 - \theta_2}\right)$$

$$g_1 = \frac{c_1 m_1 \Delta}{m_2 \theta_1 \theta_2},\ g_2 = \frac{c_1 m_1 \psi}{m_2(\theta_1 - \theta_2)}$$

$$\theta_2 = c_2 + \omega_0 + \mu_2 + \lambda_2 + \sigma_2, \tag{38}$$

$$v_1(t) = f_1 - f_2 e^{-\theta_2 t} + f_3 e^{-\theta_3 t}\ ,\ f_1 = \frac{\sigma_1 \Delta}{\theta_1\theta_2},\ f_2 = \frac{\sigma_1 \xi}{\theta_1 - \theta_2},\quad f_3 = v_1(0) - \frac{\sigma_1 \Delta}{\theta_1\theta_2} + \frac{\sigma_1 \xi}{\theta_1 - \theta_3}$$

$$v_2(t) = h_1 + h_2 e^{-\theta_1 t} + h_3 e^{-\theta_2 t} - h_4 e^{-\theta_3 t} + h_5 e^{-\theta_4 t},\ \theta_3 = \omega_0 + \mu_1 + c_1 + \alpha_1 + \gamma_1)$$

$$h_1 = \frac{1}{\theta_4}(c_1 f_1 + \sigma_2 g_1),\ h_2 = \frac{g_2 \sigma_2}{\theta_1 - \theta_4},\ h_3 = \frac{1}{\theta_2 - \theta_4}(c_1 f_2 + k),\ h_4 = \frac{c_1 f_3}{\theta_3 - \theta_4}$$

$$h_5 = v_2(0) - \frac{c_1 f_1}{\theta_4} - \frac{c_1 f_2}{\theta_2 - \theta_4} + \frac{c_1 f_3}{\theta_3 - \theta_4} - \frac{\sigma_2 g_1}{\theta_4} - \frac{\sigma_2 g_2}{\theta_1 - \theta_4} - \frac{k}{\theta_2 - \theta_4}$$

$$\theta_5 = c_1 + \mu_1 + \beta_1,\quad \theta_6 = c_2 + \mu_2 + \beta_2$$

$$i_1(t) = \frac{\lambda_1 \Delta}{\theta_1 \eta_3} + \frac{\beta_1 f_1}{\eta_3} + \frac{\lambda_1}{\theta_1 - \eta_3} e^{-\theta_1 t} + \frac{\beta_1 f_2}{(\theta_2 - \eta_3)} e^{-\theta_2 t}$$

$$+ (i_1(0) - \frac{\lambda_1 \Delta}{\theta_1 \eta_3} - \frac{\lambda_1}{\theta_1 - \eta_3} - \frac{\beta_1 f_2}{(\theta_2 - \eta_3)} - \frac{\beta_1 f_1}{\eta_3}) e^{-\eta_3 t}$$

$$i_2(t) = \frac{m_1}{m_2 \eta_4} + \frac{\lambda_2 g_1}{\eta_4} + (\frac{\lambda_2 g_2}{\theta_2 - \eta_4} - \frac{h_2 \beta_2}{\theta_1 \eta_4}) e^{-\theta_1 t} - \frac{1}{(\theta_2 - \eta_4)}(k + h_3 \beta_2) e^{-\theta_2 t} - \frac{h_4 \beta_2}{\theta_3 - \eta_4} e^{-\theta_3 t}$$

$$+ (i_2(0) - \frac{m_1}{m_2 \eta_4} - \frac{\lambda_2 g_1}{\eta_4} - \frac{g_2 \lambda_2}{\theta_1 - \eta_4} - \frac{k}{\theta_2 - \eta_4} + \frac{\beta_2 h_1}{\eta_4} + \frac{\beta_2 h_2}{\theta_1 \eta_4} - \frac{\beta_2 h}{\theta_2 - \eta_4} - \frac{\beta_2 h_4}{\theta_3 - \eta_4}) e^{-\eta_4 t},$$

$$\theta_4 = \mu_2 + \alpha_2 + c_2 + \gamma_2 + \omega_0$$

For large values of t the two disease compartments reduces to,

$$\left.\begin{aligned} i_1(t) &= \frac{\lambda_1 \Delta}{\theta_1 \eta_3} + \frac{\beta_1 f_1}{\eta_3},\ f = \frac{\sigma_1 \Delta}{\theta_1 \theta_2} \\ i_2(t) &= \frac{m_1}{m_2 \eta_4} + \frac{\lambda_2 g_1}{\eta_4},\quad g_1 = \frac{c_1 m_1 \Delta}{m_2 \theta_1 \theta_2} \end{aligned}\right\} \qquad (39)$$

Since θ_1 and θ_2 are dependent on the vaccination rate, the infective population will decay when this rate is gradually increased with time. However if we ignore vaccination of the susceptible population, we will recover the treatment scenario.

The disease-free equilibrium is,

$$E'_0 = (s_1^*, s_2^*, v_1, v_2, 0, 0, 0, 0) = (\frac{\Delta}{\theta_1}, c_1 \frac{m_1 \Delta}{m_2 \theta_1 \theta_2}, \frac{\sigma_1 \Delta}{\theta_3}, \frac{c_1 \Delta (m_1 \sigma_2 \theta_3 + \sigma_1 m_2)}{m_2 \theta_3}, 0, 0, 0, 0)$$

$$\theta_1 = (c_1 + \omega_0 + \mu_1 + \lambda_1 + \sigma_1),\quad \theta_2 = c_2 + \omega_0 + \mu_2 + \lambda_2 + \sigma_2,$$

Where θ_1 and θ_2 are dependent on the treatment and vaccination rates.

The characteristic equation at the disease-free steady state is,

$$(\eta_5 + \kappa)(\eta_6 + \kappa)\prod_{i=1}^{6}(\theta_i + \kappa) = 0$$

$$\kappa_1 = -\eta_5,\quad \kappa_2 = \eta_6,\quad \kappa_{i+2} = -\theta_i,\quad i = 1, \ldots, 6$$

Since the roots of the characteristics equation are negative, the disease-free steady state is locally asymptotically stable. Hence the infection will gradually die out when both vaccination and treatment are adopted.

The basic reproductive number in the face of vaccination and treatment is obtained using the same procedures adopted in the treatment model and so using equation (20) we obtained expression for the class dependent per capital force of vaccination for both groups as,

$$\beta_i(t) = \sum_{j=1}^{2} \pi_{ij} m_j i_j, \quad i, j = 1,2 \qquad (40)$$

Where π_{ij} the contact functions between an infective in the i group with a susceptible in the j group.

$$F = \begin{pmatrix} m_1\Delta\left(\frac{k_{11}}{\theta_1} + \frac{\pi_{11}\sigma_1}{\theta_3}\right) & m_2\Delta\left(\frac{k_{12}}{\theta_1} + \frac{\pi_{12}\sigma_2}{\theta_3}\right) & 0 & 0 \\ m_1\left(\frac{k_{21}c_1m_1\Delta}{m_2\theta_1\theta_2} + \pi_{21}\rho\right) & m_2\left(\frac{k_{22}c_1m_1\Delta}{m_2\theta_1\theta_2} + \pi_{22}\rho\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho = \frac{c_1\Delta(m_1\sigma_2\theta_3 + \sigma_1 m_2)}{m_2\theta_3}$$

The matrix V^{-1} from the treatment model is used to find the basic reproductive number.

$$FV^{-1} = \begin{pmatrix} m_1\Delta\frac{1}{\eta_3}\left(\frac{k_{11}}{\theta_1} + \frac{\pi_{11}\sigma_1}{\theta_3}\right) & m_2\Delta\frac{1}{\eta_4}\left(\frac{k_{12}}{\theta_1} + \frac{\pi_{12}\sigma_2}{\theta_3}\right) & 0 & 0 \\ m_1\frac{1}{\eta_3}\left(\frac{k_{21}c_1m_1\Delta}{m_2\theta_1\theta_2} + \pi_{21}\rho\right) & m_2\frac{1}{\eta_4}\left(\frac{k_{22}c_1m_1\Delta}{m_2\theta_1\theta_2} + \pi_{22}\rho\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The characteristic equation is,

$$\kappa^2(\kappa^2 - (A_1 + B_2)\kappa - (A_2B_1 - A_1B_2)) = 0$$

$$A_1 = m_1\Delta\frac{1}{\eta_3}\left(\frac{k_{11}}{\theta_1} + \frac{\pi_{11}\sigma_1}{\theta_3}\right), \quad A_2 = m_2\Delta\frac{1}{\eta_4}\left(\frac{k_{12}}{\theta_1} + \frac{\pi_{12}\sigma_2}{\theta_3}\right),$$

$$B_1 = m_1\frac{1}{\eta_3}\left(\frac{k_{21}c_1m_1\Delta}{m_2\theta_1\theta_2} + \pi_{21}\rho\right)$$

$$B_2 = m_2\frac{1}{\eta_4}\left(\frac{k_{22}c_1m_1\Delta}{m_2\theta_1\theta_2} + \pi_{22}\rho\right)$$

$$\kappa_1 = \frac{(A_1 + B_2)}{2} + \frac{\sqrt{(A_1 + B_2)^2 + 4(A_2B_1 - A_1B_2)}}{2}$$

$$\kappa_2 = \frac{(A_1 + B_2)}{2} - \frac{\sqrt{(A_1 + B_2)^2 + 4(A_2B_1 - A_1B_2)}}{2}$$

$$A_1 + B_2 = \frac{m_1\Delta}{\eta_2}\left(\frac{k_{11}}{\theta_1} + \frac{\pi_{11}\sigma_1}{\theta_3}\right) + \frac{m_2}{\eta_4}\left(\frac{k_{12}c_1m_1\Delta}{m_2\theta_1\theta_2} + \pi_{22}\rho\right),$$

$$A_2B_1 = \frac{m_1m_2\Delta^2}{\eta_3\eta_4}\left(\frac{k_{12}}{\theta_1} + \frac{\pi_{12}\sigma}{\theta_3}\right)\left(\frac{k_{11}}{\theta_1} + \frac{\pi_{11}\sigma}{\theta_3}\right)$$

$$A_1B_2 = \frac{m_1m_2\Delta}{\eta_3\eta_4}\left(\frac{k_{11}}{\theta_1} + \frac{\pi_{11}\sigma}{\theta_3}\right)\left(\frac{k_{12}c_1m_1\Delta}{m_2\theta_1\theta_2} + \pi_{12}\rho\right),$$

$$A_2B_1 - A_1B_2 = \frac{m_1m_2\Delta}{\eta_3\eta_4}\left(\frac{k_{11}}{\theta_1} + \frac{\pi_{11}\sigma}{\theta_3}\right)\left[\left(\Delta\left(\frac{k_{12}}{\theta_1} + \frac{\pi_{12}\sigma}{\theta_3}\right) - \left(\frac{k_{12}c_1m_1\Delta}{m_2\theta_1\theta_2} + \pi_{12}\rho\right)\right]\right.$$

$$R(\sigma) = \kappa_1$$

Since θ_1 , θ_2 are dependent on the vaccination rate and η_3 , η_4 on the removal rate for treatment, if we consider large values of θ_1 , θ_2 and η_3 , η_4, the basic reproductive number of the infection with respect to vaccination and treatment on the long run will satisfy, $1 > R(\sigma) < R_0$.

CONCLUSION

Successive increase of the vaccination and removal rates lead to increase in the value of θ_1, θ_2, η_3 and η_4 respectively and hence a corresponding decreases in the population size of the infectives for the two groups . The numbers of infective progressing to AIDS compartment will then be reduced, and the virus will disappear after some sufficiently large values of these parameters. Thus to obtain low level of transmission and possible eradication of the virus, these parameters needs to be maximized sufficiently.

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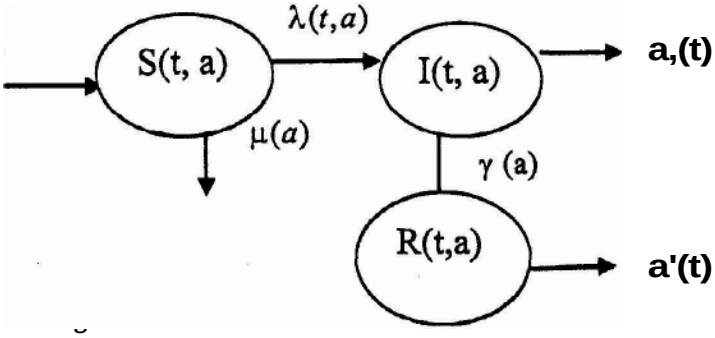
Model Transfer Diagram for each group.



Model Transfer diagram

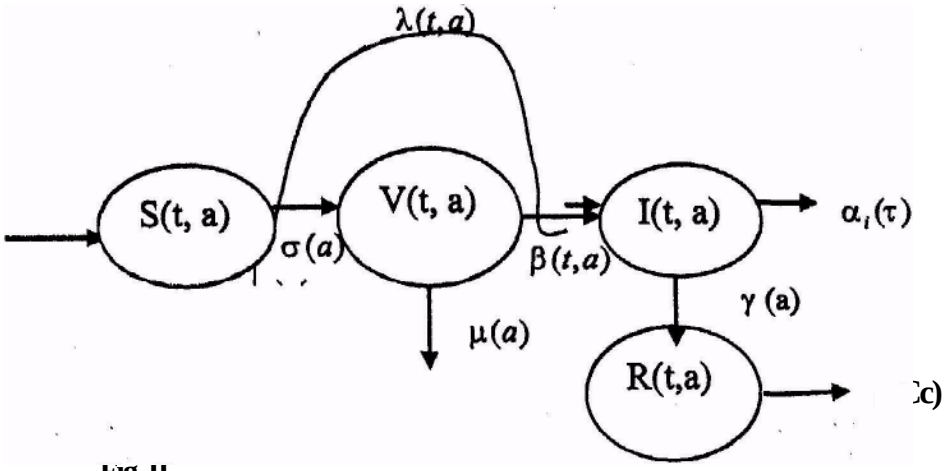


Fig. II